

Completing the classification of maximum scattered linear sets in $\text{PG}(1, q^5)$

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Abstract

We complete the classification of maximum scattered $\mathbb{F}_q$ -linear sets in $\text{PG}(1, q^5)$ up to $\text{PFL}(2, q^5)$ -equivalence. Lia, Longobardi and Zanella reduced the problem to the pseudoregulus and LP-type families together with two residual candidate families, denoted $(C3)$ and $(C4)$, previously excluded only for $q \leq 25$. We prove that neither residual family contains a maximum scattered linear set for any prime power q . Consequently, every maximum scattered $\mathbb{F}_q$ -linear set in $\text{PG}(1, q^5)$ is equivalent to a member of the pseudoregulus or LP-type family.

The proof encodes collisions as rational points on linear sections of $\text{Gr}(2, 5)$, uses a finite-extension point-count identity to pass to dual collision sections, and analyzes the resulting degree-five Pfaffian curves through their Frobenius orbits and tangent lines.

Keywords: Maximum scattered linear set, linearized polynomial, rank-metric code, Grassmannian, Pfaffian curve
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1. Introduction

Let $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^n}$. An $\mathbb{F}_q$ -linear set of rank n in $\text{PG}(1, K)$ is *maximum scattered* if each of its projective points has weight one. Equivalently, it is defined by an n -dimensional $\mathbb{F}_q$ -subspace of K^2 that meets every one-dimensional K -subspace in dimension at most one. The scattered-subspace

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bound and its finite-geometric setting go back to Blokhuis and Lavrauw [1]; projection descriptions and the first non-pseudoregulus examples appear in [2, 3]. Pseudoregulus type, equivalence, and vertex invariants were developed in [4–8]. Classifications in lower extension degrees are discussed in [9, 10]; an overview of the known families is given in [11].

Maximum scattered subspaces correspond to two-dimensional linear maximum rank-distance (MRD) codes [12–15]. Criteria based on linearized polynomials, Dickson matrices, and subresultants are available in [16–20]. These results provide the algebraic language used in degree-five classifications.

For an F -linearized polynomial f on $K = \mathbb{F}_{q^5}$, write

$$L_f = \{ \langle (x, f(x)) \rangle_K : x \in K^* \} \subset \text{PG}(1, q^5).$$

Lia, Longobardi and Zanella [21, Corollary 7.2] proved that every maximum scattered linear set in $\text{PG}(1, q^5)$ is equivalent to a member of one of four classes. Classes (C1) and (C2) are, respectively, the pseudoregulus and LP-type families [2, 14]. The two remaining candidates can be written as

$$\mathcal{C}_3(\eta, \rho) = \left\{ \left\langle \left(\eta(x^q - x) + \text{Tr}(\rho x), x^q - x^{q^4} \right) \right\rangle_{\mathbb{F}_{q^5}} : x \neq 0 \right\}, \quad (1)$$

$$\mathcal{C}_4(k) = L_{F_k}, \quad F_k(X) = k(X^q + X^{q^3}) + X^{q^2} + X^{q^4}, \quad (2)$$

where

$$\eta \neq 0, \quad \text{Tr}(\eta) = 0, \quad \text{Tr}(\rho) \neq 0, \quad \text{and} \quad N(k) = 1.$$

All traces and norms below are taken from $\mathbb{F}_{q^5}$ to $\mathbb{F}_q$. On geometric points, Frob denotes arithmetic q -Frobenius; replacing it by geometric Frobenius only reverses the cyclic indexing used below. The degree-five setting has also been studied through explicit linearized polynomials and weight distributions [22–24]. Geometric characterizations of known maximum scattered linear sets in $\text{PG}(1, q^n)$ are given in [25]. Explicit point estimates used in earlier degree-five exclusions include [26, 27]. Prior results directly relevant to the residual degree-five problem are summarized in Table 1. Lia, Longobardi and Zanella derive both residual forms in Theorem 6.3; Proposition 6.4 gives a further polynomial presentation for part of (C3). They report the $q \leq 25$ computation in Remark 6.5 and give the four-class reduction in Theorem 7.1 and Corollary 7.2 [21]. Bartoli, Giannoni, Marino and Neri characterize when their four-dimensional system U_δ is scattered in Proposition 3.6; Propositions 3.7–3.8 exclude containment in the first two LLZ classes,

while Propositions 3.9–3.11 and Theorem 3.12 establish nonextendability for $q = 3^{2h+1}$ with $h > 1$. The first open problem in Section 5 of Bartoli et al. asks whether (C3) and (C4) are empty for every q [28]. Giannoni’s constructions concern the distinct $(m - 3)$ -scattered regime [29].

Table 1: Status of the residual degree-five classification.

Reference	Scope	Milestone	Residual status
Montanucci–Zanella [23]	A two-parameter class	Canonical forms	Unresolved
Lia et al. [21]	All q	Four-class reduction	Open for $q > 25$
Bartoli et al. [28]	All q	Containment tests	Open
This paper	All q	All-q exclusion	Complete

Theorem 1.1. *For every prime power q , every maximum scattered $\mathbb{F}_q$ -linear set in $\text{PG}(1, q^5)$ is equivalent, under $\text{P}\Gamma\text{L}(2, q^5)$, to one of*

$$L_{X^q} \quad \text{or} \quad L_{X^{q^s} + \delta X^{q^5-s}}, \quad s \in \{1, 2\}, \quad N(\delta) \notin \{0, 1\}.$$

These are the pseudoregulus and LP-type families. No member of either residual family (C3) or (C4) is maximum scattered.

The following formal consequence combines the classification with the containment exclusions of Bartoli, Giannoni, Marino and Neri [28]. For $\delta \in K$ with $N(\delta) = 1$, put

$$U_\delta = \{(x, x^q + \delta x^{q^4}) : \text{Tr}(x) = 0\} \subset K^2.$$

Corollary 1.2. *If U_δ is scattered, then it is maximal with respect to inclusion among scattered F -subspaces of K^2 . Equivalently, its associated $[4, 2, 3]_{q^5/q}$ MRD code is nonextendable.*

Proof. By the scattered-subspace bound, every proper scattered extension of U_δ has F -dimension five and is therefore maximum scattered. The containment argument requires the classification at the level of defining subspaces.

Theorem 3.3 of Bartoli et al. [28] states that, up to $\Gamma L(2, K)$, every five-dimensional maximum scattered subspace of K^2 has one of the following forms:

$$\text{PR}_s = \{(x, x^{q^s}) : x \in K\}, \quad 1 \leq s \leq 4,$$

$$\text{LP}_{s,\eta} = \{(x, x^{q^s} + \eta x^{q^{5-s}}) : x \in K\}, \quad s \in \{1, 2\}, \quad N(\eta) \notin \{0, 1\},$$

or a defining subspace from $(C3)$ or $(C4)$. Thus the list includes all four pseudoregulus exponents and both LP exponents, together with their ΓL -orbits. Proposition 3.7 of [28] excludes containment of U_δ , up to $\Gamma L(2, K)$, in every PR_s on this list. Proposition 3.8 of the same paper excludes containment in every $\text{LP}_{s,\eta}$ on the list. Finally, Propositions 5.1 and 5.2 show that the remaining two types contain no maximum scattered member. Hence no proper scattered extension of U_δ exists.

Since a scattered four-dimensional U_δ cannot lie in a K -line, it spans K^2 and is a $[4, 2, 3]_{q^5/q}$ q -system by [28, Theorem 2.8]. Its maximality is equivalent to nonextendability of the associated MRD code by [28, Corollary 2.11]. $\square$

The proof associates two linear sections C and D with a rank-five subspace. The section C parametrizes two-dimensional collision spaces, and a finite incidence count gives

$$\#C(E) = \#D(E)$$

over every finite extension $E/\mathbb{F}_q$. In each of the two collision models used in the proof, the dual section is a height-three Pfaffian curve of degree five and arithmetic genus one. For a point-free section, the component through each point of a distinguished Frobenius orbit is the corresponding tangent line. Evaluation on those lines classifies the scattered trace perturbations $X^q - c \text{Tr}(X)$ and excludes the trace-zero extensions $\mathcal{W}_n$ of the punctured Gabidulin graph. The two LLZ families reduce to these models. At the nonzero scattered trace parameters, the dual collision section is a Frobenius-twisted pentagon, while the primal section has the same point counts over every finite extension; see Corollary 4.2.

The point-count equality is the finite-field point-count realization of the Grothendieck-ring incidence relation in [30, Proposition 2.3 and the proof of Theorem 2.9]. The direct rank-stratified proof below also covers characteristic two and sections without smoothness or expected-dimension hypotheses. The height-three resolution is due to Buchsbaum and Eisenbud [31], while Fisher [32] treats Pfaffian genus-one models. Hulek [33] discusses elliptic normal curves and their degree-five degenerations.

For each collision model, we apply these tools to its explicit dual section; the tangent restrictions yield the reductions of $(C3)$ and $(C4)$.

Section 2 proves the primal–dual point-count identity, and Section 3 gives the Pfaffian tangent criterion. Section 4 treats the two collision models used in Section 5 to exclude $(C3)$ and $(C4)$. The model-specific calculations appear in Appendices A and B.

2. Collision sections and the dual point count

Let $F = \mathbb{F}_q$, let $K = \mathbb{F}_{q^5}$, and let V be a five-dimensional F -space. For an injective F -linear map $L : V \rightarrow K^2$, define

$$\Phi_L : \bigwedge_F^2 V \longrightarrow \bigwedge_K^2 K^2 \simeq K, \quad v \wedge w \longmapsto \det_K(L(v), L(w)). \quad (3)$$

When Φ_L has rank five, put

$$C_L = \mathrm{Gr}(2, V) \cap \mathbf{P}(\ker \Phi_L) \subset \mathbf{P}(\bigwedge^2 V), \quad (4)$$

and, for $A = \mathrm{im}(\Phi_L^*) \subset \bigwedge^2 V^*$,

$$D_L = \mathrm{Gr}(2, V^*) \cap \mathbf{P}(A). \quad (5)$$

Lemma 2.1. *$L(V)$ is scattered if and only if $C_L(F) = \emptyset$.*

Proof. An F -point of $\mathrm{Gr}(2, V)$ is a two-dimensional F -space $U = \langle v, w \rangle_F$. It belongs to C_L precisely when $\det_K(L(v), L(w)) = 0$, or equivalently when $L(U)$ lies in one K -line. This is a point of weight at least two in the linear set defined by $L(V)$. $\square$

Lemma 2.2 (invariance of scatteredness). *Let $L : V \rightarrow K^2$ be injective.*

- (i) *If $A \in \mathrm{GL}_2(K)$, $B \in \mathrm{GL}_F(V)$, and $L' = A \circ L \circ B$, then $\bigwedge^2 B$ induces an isomorphism $C_{L'} \simeq C_L$. The map B is a reparametrization of the defining subspace. The map A gives a $\mathrm{GL}_2(K)$ -equivalence of defining subspaces and a projective equivalence of their linear sets.*
- (ii) *Let $f : K \rightarrow K$ be F -linear and let f^* be its adjoint for the trace pairing, defined by*

$$T(f(x)y) = T(xf^*(y)).$$

The graph of f is scattered if and only if the graph of f^ is scattered.*

(iii) If $\alpha, \beta \in K^*$ and $g(x) = \beta f(\alpha x)$, then

$$U_g = \text{diag}(\alpha^{-1}, \beta) U_f, \quad U_f = \{(x, f(x)) : x \in K\}.$$

Coordinate swaps and shears are covered by part (i).

Proof. For part (i) and $v, w \in V$,

$$\Phi_{L'}(v \wedge w) = \det(A) \Phi_L(Bv \wedge Bw).$$

It follows that $\ker \Phi_{L'} = (\bigwedge^2 B)^{-1}(\ker \Phi_L)$, which gives the claimed isomorphism of collision sections. The assertion about A also follows directly because it maps K -lines to K -lines.

For part (ii), the graph of f is scattered precisely when

$$\dim_F \ker(f - mI) \leq 1 \quad (m \in K).$$

Multiplication by m is self-adjoint for $T(xy)$, and an operator and its adjoint have the same rank. Hence $f - mI$ and $f^* - mI$ have the same nullity for every m . Part (iii) follows by setting $y = \alpha x$ in the graph parametrization. $\square$

Parts (i) and (iii) give explicit equivalences of defining subspaces. Part (ii) will be used only to transfer the scatteredness condition.

The required primal–dual equality follows from the next incidence count. Its rank-stratified proof also covers characteristic two and sections of non-expected dimension.

Lemma 2.3 (primal–dual point equality). *Let $A \subset \bigwedge^2 V^*$ have dimension five and put*

$$C = \text{Gr}(2, V) \cap \mathbf{P}(A^\perp), \quad D = \text{Gr}(2, V^*) \cap \mathbf{P}(A).$$

Then, for every finite extension F_Q/F , where $Q = q^r$ for some $r \geq 1$,

$$\#C(F_Q) = \#D(F_Q).$$

Proof. Write $G = \# \text{Gr}(2, 5)(F_Q)$ and $P_j = \#\mathbf{P}^j(F_Q)$. Count

$$\mathcal{J} = \{(U, [\omega]) \in \text{Gr}(2, V)(F_Q) \times \mathbf{P}(A)(F_Q) : \omega|_U = 0\}.$$

For fixed U , the fibre is $\mathbf{P}^4$ if $U \in C$ and $\mathbf{P}^3$ otherwise, so

$$\#\mathcal{J} = GP_3 + Q^4 \#C(F_Q). \tag{6}$$

A nonzero alternating form on a five-space has rank two or four, also in characteristic two. For rank four, its radical is a line. There are P_3 isotropic two-spaces containing the radical. Those that do not contain it project to Lagrangian planes in a symplectic four-space. There are $(Q+1)(Q^2+1)$ such planes, each with Q^2 lifts. Therefore

$$I_4 = P_3 + Q^2(Q+1)(Q^2+1). \quad (7)$$

For rank two, the radical is a three-space. A two-space is nonisotropic precisely when it is the graph of a map from the two-dimensional symplectic quotient to the radical. Hence

$$I_2 = G - Q^6. \quad (8)$$

Equations (7) and (8) give

$$I_2 - I_4 = Q^4, \quad P_4 I_4 = G P_3. \quad (9)$$

The rank-two alternating forms are the decomposable forms and therefore the points of D . Counting $\mathcal{J}$ by its second projection gives

$$\#\mathcal{J} = P_4 I_4 + Q^4 \#D(F_Q) = G P_3 + Q^4 \#D(F_Q).$$

Comparison with (6) gives $\#C(F_Q) = \#D(F_Q)$. $\square$

3. The Pfaffian tangent criterion

The criterion below also applies when the Pfaffian scheme is reducible or nonreduced.

Lemma 3.1. *Let $X \subset \mathbf{P}_F^4$ be defined by the five submaximal Pfaffians of a 5×5 alternating matrix of linear forms, and assume that its Pfaffian ideal has height three. Choose projective coordinates on $\mathbf{P}_{\overline{F}}^4$, not necessarily descending to F , and denote their five coordinate vertices by $e_0, \dots, e_4$. Suppose that these vertices are smooth points of $X_{\overline{F}}$ and that Frobenius permutes them cyclically. If $X(F) = \emptyset$, then the geometric irreducible component through e_0 , endowed with its reduced induced structure, is the projective tangent line to X at e_0 .*

Proof. Put $R = \overline{F}[x_0, \dots, x_4]$ and let $I_X \subset R$ be the Pfaffian ideal. Faithful flatness of $\overline{F}/F$ preserves the height. Since R is Cohen–Macaulay, the height-three assumption is the grade condition in the Buchsbaum–Eisenbud acyclicity theorem [31, 32]. The Pfaffian complex is therefore exact:

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R/I_X \longrightarrow 0. \quad (10)$$

All entries in its three nonidentity differentials have positive degree, so the resolution is minimal. Its Hilbert series is

$$\frac{1 - 5z^2 + 5z^3 - z^5}{(1 - z)^5} = \frac{1 + 3z + z^2}{(1 - z)^2}.$$

Thus R/I_X is a two-dimensional Cohen–Macaulay Gorenstein ring with Hilbert polynomial $5m$. It follows that $X_{\overline{F}}$ is a pure arithmetically Gorenstein curve of degree five and arithmetic genus one. Cohen–Macaulayness also rules out embedded and isolated components. This use of the Pfaffian complex is valid in characteristic two: alternating means that the diagonal entries vanish, and the complex is defined over the integers.

Smoothness at e_0 places it on a unique geometric irreducible component Γ , which is taken with its reduced induced structure. Since Frobenius⁵ fixes e_0 , the Frobenius orbit of Γ has length one or five.

If it has length five, the five conjugate components are distinct and each has positive degree. The degree of a pure curve is the sum of the degrees of its geometric components counted with their generic multiplicities. Since $\deg X = 5$, all five components have degree one and generic multiplicity one, and no further one-dimensional component occurs. Cohen–Macaulayness excludes embedded components. Thus Γ is a line; at the smooth point e_0 , it is the projective tangent line.

Suppose that Γ is Frobenius-stable. Its Frobenius-invariant homogeneous prime ideal descends to F , giving a curve Γ_0 over F with $\Gamma_0 \times_F \overline{F} = \Gamma$. Thus Γ_0 is geometrically integral. Let $\tilde{\Gamma}_0$ be its normalization. This is a regular projective curve and hence is smooth over the perfect field F . Therefore $C = \tilde{\Gamma}_0 \times_F \overline{F}$ is normal, and its finite birational map to Γ identifies C with the normalization of Γ . This also proves that normalization is compatible here with the separable base change $\overline{F}/F$.

The component contains all five e_i and is therefore nondegenerate in $\mathbf{P}^4$, so $d = \deg \Gamma$ belongs to $\{4, 5\}$. Let $\nu : C \rightarrow \Gamma$ be the normalization map and put $L = \nu^* \mathcal{O}_\Gamma(1)$. Then $\deg L = d$. The five coordinate linear forms restrict to linearly independent sections on Γ , and their pullbacks remain independent, so $h^0(C, L) \geq 5$. If $d = 4$ and $g(C) > 0$, Riemann–Roch gives $h^0(C, L) = 5 - g(C) \leq 4$ when L is nonspecial, whereas Clifford’s theorem gives $h^0(C, L) \leq 3$ when L is special. Hence $g(C) = 0$. If $d = 5$ and $g(C) \geq 2$, the same alternatives give $h^0(C, L) = 6 - g(C) \leq 4$ or $h^0(C, L) \leq 3$, again a contradiction. Thus $g(C) \leq 1$.

Over a finite field, such a curve has an F -point by the genus-zero case and the Hasse–Weil bound [34, 35]. Its image belongs to $\Gamma_0(F) \subset X(F)$, a contradiction. $\square$

For the two models in Section 4, height three will be verified by the following coordinate-plane criterion.

Lemma 3.2 (coordinate-plane criterion). *Let $X = \text{Gr}(2, 5) \cap \mathbf{P}^4 \subset \mathbf{P}^9$ be a linear section, written in its dual $\mathbf{P}^4$ by the submaximal Pfaffians of a 5×5 alternating matrix. Suppose that $e_0, \dots, e_4 \in X_{\overline{F}}$ and that the Jacobian of the five Pfaffians has rank three at each vertex. If a projective plane meets $X_{\overline{F}}$ set-theoretically only in three of these vertices, then the Pfaffian ideal has height three, and all five vertices are smooth points of the resulting curve.*

Proof. Here $X_{\overline{F}}$ is the intersection of the six-dimensional Grassmannian $\text{Gr}(2, 5)_{\overline{F}}$ with a four-plane in $\mathbf{P}_{\overline{F}}^9$. The projective dimension theorem shows that every irreducible component has dimension at least $6 + 4 - 9 = 1$. At a coordinate vertex, Jacobian rank three gives a one-dimensional projective Zariski tangent space. Every component has dimension at least one, so the local ring has dimension and embedding dimension one. Each vertex is therefore a regular curve point.

If a component had dimension at least two, the projective dimension theorem would force it to meet the chosen plane. Such an intersection point would be one of the three specified vertices, where the local dimension is one, a contradiction. Every component consequently has dimension one, and the Pfaffian ideal has height three, the maximal height for the submaximal Pfaffian ideal of a 5×5 alternating matrix [31, Theorem 2.1]. $\square$

For the converse in Theorem 4.1, the same degree-five geometry gives the following five-line description.

Lemma 3.3 (Frobenius pentagon). *Let $X \subset \mathbf{P}_{\overline{F}}^4$ be defined by the five submaximal Pfaffians of a 5×5 alternating matrix of linear forms, and assume that its Pfaffian ideal has height three. Choose projective coordinates on $\mathbf{P}_{\overline{F}}^4$ whose coordinate vertices $e_0, \dots, e_4$ are smooth points of $X_{\overline{F}}$ and are permuted cyclically by Frobenius. Suppose that a projective line L_0 , contained in $X_{\overline{F}}$ as a closed subscheme, contains e_0 , that $\text{Frobenius}^5(L_0) = L_0$, and that*

$$L_0 \cap \text{Frob}(L_0) = \emptyset.$$

Put $L_i = \text{Frob}^i(L_0)$. Then $X_{\overline{F}}$ is a reduced union of the five lines $L_0, \dots, L_4$, and $X(F) = \emptyset$. Its five ordinary nodes form one Frobenius orbit, and the line-incidence graph is a five-cycle. In particular, for every $r \geq 1$,

$$\#X(\mathbb{F}_{q^r}) = \begin{cases} 0, & 5 \nmid r, \\ 5q^r, & 5 \mid r. \end{cases} \quad (11)$$

Consequently

$$Z(X/F, u) = \frac{1}{1 - q^5 u^5}. \quad (12)$$

No assumption on $X(F)$ is made in the hypotheses.

Proof. The purity, degree, arithmetic genus, and Cohen–Macaulay properties used below follow from the Pfaffian resolution (10); their derivation does not use the existence or absence of F -points. The Frobenius orbit length of L_0 divides five. The disjointness of L_0 and $\text{Frob}(L_0)$ excludes orbit length one, so the five lines L_i are distinct. By (10), X is pure of degree five. Each line contributes its scheme-theoretic generic multiplicity to the degree. The five lines therefore exhaust the geometric support and all have generic multiplicity one. The same resolution makes X Cohen–Macaulay, hence it satisfies Serre’s condition S_1 . At the generic point of each supporting line, multiplicity one says that the zero-dimensional local ring has length one; it is therefore a field. Thus $X_{\overline{F}}$ satisfies R_0 , and the criterion $R_0 + S_1$ shows that it is reduced.

We next determine the rational points over F . If $P \in X(F)$ and $P \in L_i$, then for every j ,

$$P = \text{Frob}^j(P) \in \text{Frob}^j(L_i) = L_{i+j}.$$

In particular, P belongs to two consecutive lines. This contradicts $L_i \cap L_{i+1} = \emptyset$, obtained by applying Frobenius to the disjointness hypothesis. Hence $X(F) = \emptyset$.

Figure 1 displays the Frobenius orbit of the line components separately from their component-incidence cycle.

After sheafifying the resolution, the vanishing of the intermediate cohomology of the indicated line bundles on $\mathbf{P}^4$ gives $h^0(X_{\overline{F}}, \mathcal{O}_X) = 1$. Hence the union is connected. Its normalization is the disjoint union of five projective lines. Since $p_a(X) = 1$, the normalization sequence gives

$$\sum_{x \in \text{Sing}(X_{\overline{F}})} \delta_x = \chi(\mathcal{O}_{\tilde{X}}) - \chi(\mathcal{O}_X) = 5.$$

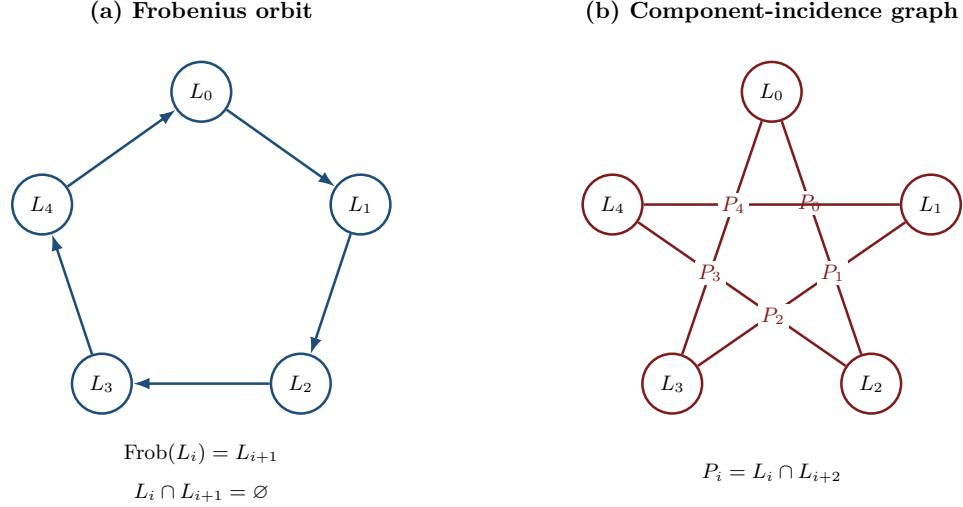


Figure 1: The two cyclic structures in Lemma 3.3. In the left panel, Frobenius sends L_i to L_{i+1} and successive conjugates are disjoint. The right panel is the component-incidence graph, whose edge P_i represents the node $L_i \cap L_{i+2}$; it is an incidence schematic, not a projective-plane realization of the five line components. Indices are taken modulo five.

Every intersection point of two components is fixed by Frobenius⁵, so its Frobenius orbit has length one or five. An orbit of length one would give an F -point of X , which has just been excluded. Connectedness guarantees that some components intersect. Hence the singular locus contains an orbit of five points. The displayed delta sum forces this to be the only orbit and gives $\delta_x = 1$ at every point. Every branch is a smooth line branch. If r_x is the number of branches, the normalization sequence gives $\delta_x \geq r_x - 1$. A point lying on only one line is smooth, so $\delta_x = 1$ forces exactly two branches. They belong to distinct lines and have distinct tangent directions; the singularity is therefore an ordinary node. Frobenius is transitive on both the five lines and the five nodes, so every vertex of the incidence graph has the same degree. Its degree sum is ten; connectedness then identifies it as a five-cycle. Frobenius invariance makes the edge difference constant modulo five, equal to $\{\pm 1\}$ or $\{\pm 2\}$. The disjointness of consecutive lines excludes $\{\pm 1\}$, so the nodes are $P_i = L_i \cap L_{i+2}$.

If $5 \mid r$, all five lines and all five nodes are defined over $\mathbb{F}_{q^r}$, and inclusion-exclusion gives $5(q^r + 1) - 5 = 5q^r$ points. If $5 \nmid r$, an $\mathbb{F}_{q^r}$ -point would have to lie on two Frobenius-conjugate components and hence in the five-element

orbit of nodes, which has no point fixed by Frobenius^r. This proves (11). Substitution in the definition of the zeta function gives

$$\exp\left(\sum_{m \geq 1} \frac{5q^{5m}u^{5m}}{5m}\right) = (1 - q^5u^5)^{-1}.$$

□

4. Trace and vertical collision criteria

4.1. Trace perturbations

Theorem 4.1 (trace perturbations). *Let $K = \mathbb{F}_{q^5}$ and $T = \text{Tr}_{K/F}$. For $c \in K$, set*

$$f_c(X) = X^q - cT(X).$$

Then f_c is scattered if and only if either $c = 0$, or

$$c \in F^* \quad \text{and} \quad 7c^2 - 5c + 1 = 0.$$

Proof. The polynomial $f_0(X) = X^q$ is scattered. Assume from now on that $c \neq 0$.

For necessity, base-change to $\bar{F}$ and use the five embeddings of K/F . Let $e_0^*, \dots, e_4^*$ be the coordinate covectors, let $S^* = \sum_j e_j^*$, and write $c_i = c^{q^i}$, with indices modulo five. Index the factors so that arithmetic Frobenius sends the i th factor to the $(i+1)$ st. It then sends c_i to c_{i+1} and cyclically permutes the coordinate vertices. For the graph $x \mapsto (x, f_c(x))$, the coordinate determinant forms are obtained as follows. Write $x_i = e_i^*(x)$. In the i th split factor, $T(x) = \sum_j x_j$, the first graph coordinate is x_i , and the second is $x_{i+1} - c_i \sum_j x_j$. The i th coordinate of the determinant of two graph vectors is therefore the evaluation of the two-form

$$\ell_i = e_i^* \wedge (e_{i+1}^* - c_i S^*). \tag{13}$$

If $\sum_i t_i \ell_i = 0$, the nonadjacent entries give a common value for the products $t_i c_i$, while an adjacent entry gives $t_i = 0$ for every i . Thus the determinant map in Lemma 2.1 has rank five.

Let $D_c \subset \mathbf{P}_{t_0, \dots, t_4}^4$ be the rank-two locus of $\sum_i t_i \ell_i$. The five Pfaffians are listed in Appendix A.1. They show that every coordinate vertex e_i lies on D_c and that the Jacobian has a nonzero 3×3 minor, $c_0^2 c_4$, at e_0 .

Frobenius gives the same lower bound elsewhere, while the Pfaffian bound gives height at most three. On the coordinate plane $t_0 = t_1 = 0$, the relevant restrictions are

$$Q_2 = -c_3 t_3 t_4, \quad Q_3 = -c_2 t_2 t_4, \quad Q_0 = c_2 t_2 t_3 + c_4 t_2 t_4.$$

Their common zero set is $\{e_2, e_3, e_4\}$, so Lemma 3.2 gives height three.

Assume that f_c is scattered. Then $D_c(F) = \emptyset$ by Lemmas 2.1 and 2.3, and Lemma 3.1 identifies the tangent line at e_0 as a component of D_c .

The tangent calculation in Appendix A.1 has three cases. If $c_0 + c_3 = 1$, the tangent direction satisfies $Q_1(v) = -c_0^2/c_2 \neq 0$. If $c_0 + c_3 \neq 1$ and $c_0 + c_2 + c_3 - 1 = 0$, it satisfies $Q_3(v) = -c_4 \neq 0$. Both cases contradict containment of the tangent line. In the remaining case,

$$Q_4(v) = -\frac{c_4^2(c_1 - c_3)(c_0 + c_2 + c_3 - 1)}{c_0(c_0 + c_3 - 1)^2}. \quad (14)$$

Containment forces $c_1 = c_3$, that is, $(c^q)^{q^2} = c^q$. Thus $c^q \in \mathbb{F}_{q^2} \cap K = F$, and applying the inverse Frobenius automorphism gives $c \in F$.

For scalar c , the remaining evaluations are

$$\begin{aligned} Q_1(v) &= -\frac{c(7c^2 - 5c + 1)}{(2c - 1)^2}, \quad Q_2(v) = \frac{c(7c^2 - 5c + 1)}{(2c - 1)^2}, \\ Q_3(v) &= \frac{7c^2 - 5c + 1}{2c - 1}. \end{aligned} \quad (15)$$

The exceptional value $2c = 1$ was excluded above, so $7c^2 - 5c + 1 = 0$.

Conversely, suppose that $c \in F^*$ and $7c^2 - 5c + 1 = 0$. The equality $2c = 1$ is impossible: in odd characteristic it would give $4(7c^2 - 5c + 1) = 1$, and in characteristic two its left-hand side is zero. Put $d = 2c - 1$. The tangent direction (A.3) becomes

$$v_0 = \left(0, \frac{3c - 1}{d}, -\frac{c}{d}, \frac{c}{d}, 1\right). \quad (16)$$

The scalar evaluations in (15), together with $Q_0(v_0) = Q_4(v_0) = 0$, show that every Pfaffian vanishes on v_0 . For a homogeneous quadratic form Q , tangency at e_0 means that the polar term $B_Q(e_0, v_0)$ vanishes. Since $Q(e_0) = Q(v_0) = 0$, it follows that $Q(ae_0 + bv_0) = 0$ for all $a, b \in \overline{F}$. Hence every Pfaffian vanishes on the line

$$L_0 = \langle e_0, v_0 \rangle_{\overline{F}} \subset D_c.$$

Frobenius cyclically shifts the coordinates and fixes c . If $L_1 = \text{Frob}(L_0)$, the minor on rows 0, 1, 2, 3 of the four column vectors spanning L_0 and L_1 is

$$\det(e_0, v_0, e_1, \text{Frob}(v_0))_{0,1,2,3} = \frac{c}{2c-1} \neq 0. \quad (17)$$

Hence $L_0 \cap L_1 = \emptyset$. Lemma 3.3 gives $D_c(F) = \emptyset$. By Lemmas 2.1 and 2.3, the graph of f_c is scattered. $\square$

Corollary 4.2 (collision zeta function). *Suppose that $c \in F^*$ and $7c^2 - 5c + 1 = 0$. Let C_{f_c} and D_c be the primal and dual collision sections of the graph of f_c . Then, for every $r \geq 1$,*

$$\#C_{f_c}(\mathbb{F}_{q^r}) = \#D_c(\mathbb{F}_{q^r}) = \begin{cases} 0, & 5 \nmid r, \\ 5q^r, & 5 \mid r, \end{cases}$$

and both sections have zeta function $(1 - q^5 u^5)^{-1}$.

Proof. The description of D_c follows from the converse part of Theorem 4.1 and Lemma 3.3. The equality for C_{f_c} follows from Lemma 2.3 over every finite extension. $\square$

Corollary 4.3 (LP-type identification). *Suppose that $c \in F^*$ and $7c^2 - 5c + 1 = 0$, and put*

$$\delta = \frac{c}{1-2c}.$$

Then $\delta^2 - \delta + 1 = 0$ and $N(\delta) = 1 - \delta \notin \{0, 1\}$. The graph of $f_c = \tau - cT$ is linearly equivalent to the graph of

$$g_\delta = \tau^2 + \delta\tau^3.$$

In particular, the nonzero scattered trace perturbations are of LP type with exponent $s = 2$.

Proof. The denominator is nonzero: in odd characteristic the equality $2c = 1$ would give $4(7c^2 - 5c + 1) = 1$, while in characteristic two it equals one. Direct substitution gives

$$(1-2c)^2(\delta^2 - \delta + 1) = 7c^2 - 5c + 1, \quad \delta(1-\delta) = 1.$$

The quadratic relation gives $\delta^3 = -1$ and hence $N(\delta) = \delta^5 = 1 - \delta$. The second displayed identity shows that δ and $1 - \delta$ are nonzero; moreover $1 - \delta \neq 1$.

This verifies the LP norm condition. In the algebra $F[\tau]/(\tau^5 - I)$, the same quadratic relation gives

$$-\delta(f_c + I) = g_\delta(f_c + \delta I). \quad (18)$$

Consequently the matrix

$$A_\delta = \begin{pmatrix} \delta & 1 \\ -\delta & -\delta \end{pmatrix}, \quad \det A_\delta = 1,$$

sends $(x, f_c(x))$ to $(y, g_\delta(y))$, where $y = (f_c + \delta I)(x)$. Since A_δ and the graph parametrization are injective, $x \mapsto y$ is bijective. Hence $A_\delta(U_{f_c}) = U_{g_\delta}$. $\square$

Remark 4.4. *The theorem also counts the nonmonomial trace perturbations. If $q = 2^e$, there are two when e is even and none when e is odd. In characteristics three and seven there is one solution in each case, namely $c = 1$ and $c = 3$, respectively. In every other characteristic, there are two precisely when -3 is a square in F , and none otherwise.*

4.2. Trace-zero vertical extensions

The graph

$$\{(h, h^{q^4}) : h \in H\}$$

is the restriction of a Gabidulin graph to the trace-zero hyperplane and is therefore a punctured Gabidulin graph. Rank-metric puncturing is treated in [36]; related extension constructions are studied in [37]. The spaces in the following theorem are its one-dimensional vertical extensions.

Theorem 4.5 (trace-zero vertical extensions). *Let $H = \ker T$ and let $n \in K^*$ satisfy $T(n) = 0$. Then*

$$\mathcal{W}_n = \{(h, h^{q^4} + tn) : h \in H, t \in F\} \subset K^2$$

is not scattered.

Proof. Let $V = H \oplus F$, where $\oplus$ denotes an external direct sum. In characteristic five, $F \subset H$ inside K , so this is not an internal decomposition of K . Over $\overline{F}$, let $r_0, \dots, r_4$ be the restrictions to H of the five embedding covectors, so $\sum_i r_i = 0$, and let s extract the last F -coordinate. Put $n_i = n^{q^i}$. For

$$(h, t) \longmapsto (h, h^{q^4} + tn),$$

the determinant forms follow directly from the split coordinates. The i th embedding sends the first component to $r_i(h)$ and the second to $r_{i-1}(h) + n_i t$, since $(h^{q^4})^{q^i} = h^{q^{i-1}}$. With $s(h, t) = t$, the i th determinant coordinate is represented by

$$\ell_i = r_i \wedge (r_{i-1} + n_i s). \quad (19)$$

If $\sum_i z_i \ell_i = 0$, its s -part gives a common value for the products $z_i n_i$. After substituting $r_4 = -r_0 - r_1 - r_2 - r_3$, the coefficient of $r_0 \wedge r_2$ gives $z_0 = 0$, and then $z_i = 0$ for every i . Thus the five determinant forms are independent.

The dual locus $D_n \subset \mathbf{P}_{z_0, \dots, z_4}^4$ and its Pfaffians are recorded in Appendix A.2. The coordinate vertices form a Frobenius orbit. At e_0 the Jacobian has a nonzero 3×3 minor, $n_0^2 n_1$; Frobenius and the general Pfaffian height bound show that its rank is three at every vertex. On the plane $z_0 = z_1 = 0$, the Pfaffians cut out precisely e_2, e_3, e_4 : the relation $Q_4 = z_2 z_4$ gives the two cases, after which Q_1 or Q_0 removes the remaining coordinate. Hence Lemma 3.2 gives height three.

Set $\delta = n_0 + n_2 + n_3$. This element is nonzero. If $\delta = 0$, then $\sum n_i = 0$ gives $n_1 + n_4 = 0$, while the Frobenius image of $\delta = 0$ is $n_1 + n_3 + n_4 = 0$, forcing $n_3 = 0$.

If $\mathcal{W}_n$ were scattered, the dual section would have no F -point, so Lemma 3.1 would make its tangent line at e_0 a component. The tangent calculation in Appendix A.2 gives

$$Q_1(v) = \frac{(n_2 - n_4)\delta}{n_0}. \quad (20)$$

It follows that $n_2 = n_4$, so $(n^{q^2})^{q^2} = n^{q^2}$. Hence $n^{q^2} \in \mathbb{F}_{q^2} \cap K = F$, and $n = (n^{q^2})^{q^3} \in F$. If $\text{char } F \neq 5$, the relation $T(n) = 5n = 0$ contradicts $n \neq 0$. In characteristic five, the tangent direction is, up to scale,

$$v = (0, 2, 1, -1, 3),$$

but $Q_4(v) = 7 = 2 \neq 0$. Thus $\mathcal{W}_n$ is not scattered. $\square$

5. Application to the LLZ candidates

The reductions below use the three operations separated in Lemma 2.2: ambient $\text{GL}_2(K)$ transformations, F -linear reparametrizations, and trace adjunction. The first two give equivalent defining subspaces. Trace adjunction is used only to transfer scatteredness. The cases and their concluding contradictions are listed in Table 2.

Table 2: Cases in the exclusion of the two candidate families.

Family	Case	Reduced object	Concluding contradiction
(C3)	$T(u) \neq 0$	$x^q - cT(x)$	Incompatible trace conditions ($c \in F$).
(C3)	$T(u) = 0$	$\mathcal{W}_n, T(n) = 0$	Excluded by Theorem 4.5.
(C4)	$k = 1$	$F_1 _{\ker T} = -I$	A four-dimensional collision space.
(C4), odd	$D \neq 0$	$x^q - cT(x)$	$k^3 = k^5 = 1$, contrary to $k \neq 1$.
(C4), odd	$D = 0$	Adjoint graph $x^q - nT(x)$	Characteristic 11 remains, but $P(1/5) \neq 0$.
(C4), char. 2	$D \neq 0$	$x^q - cT(x)$	$k^3 = k^5 = 1$, contrary to $k \neq 1$.
(C4), char. 2	$D = 0$	Adjoint graph $x^q - nT(x)$	$T(n) = 1$ gives $n = 1$, while $P(1) = 1$.

For (C3), $u = \eta^{-1}$. For (C4), $P(X) = 7X^2 - 5X + 1$ and $s = (k-1)^{-1}$; in odd characteristic $D = 2 + T(s)$, while in characteristic two $D = T(s)$.

5.1. The (C3) family

Proposition 5.1. *No member of the family (C3) is maximum scattered.*

Proof. Set

$$y = x^q - x \in H, \quad t = T(\rho x) \in F.$$

The map $x \mapsto (y, t)$ is injective: if $y = 0$, then $x \in F$, and $t = xT(\rho) = 0$ gives $x = 0$. Its domain and codomain have the same F -dimension, so it is bijective. Since $x^q - x^{q^4} = y + y^{q^4}$, this bijective F -linear reparametrization makes the defining subspace in (1) equal to

$$U_\eta = \{(\eta y + t, y + y^{q^4}) : y \in H, t \in F\}. \quad (21)$$

To check its rank, assume $y + y^{q^4} = 0$. This gives $y^{q^2} = y$, so $y \in \mathbb{F}_{q^2} \cap K = F$. If the characteristic is different from five, $T(y) = 5y = 0$ gives $y = 0$. In characteristic five, the relation $y = -y^q$ and $y \in F$ give $2y = 0$, with the same conclusion. The first coordinate then gives $t = 0$.

Let $u = \eta^{-1}$. The change of coordinates

$$(X, Y) \mapsto (Y - uX, uX)$$

followed by the substitution $h = y^{q^4}$ sends U_η to

$$U_u = \{(h - ut, h^q + ut) : h \in H, t \in F\}. \quad (22)$$

The coordinate map has determinant $-u$ and lies in $\mathrm{GL}_2(K)$; $y \mapsto h$ is an F -linear reparametrization of H .

Case $T(u) \neq 0$. Set $a = T(u)$ and $x = h - ut$. Then $t = -T(x)/a$, and (22) is the graph of

$$x \mapsto x^q - \frac{u + u^q}{a}T(x) = x^q - cT(x). \quad (23)$$

With $b = u/a$,

$$c = b + b^q, \quad T(b) = 1, \quad T(b^{-1}) = 0. \quad (24)$$

If the graph were scattered, then $c \in F$: this is immediate for $c = 0$ and follows from Theorem 4.1 for $c \neq 0$. Comparing (24) with its Frobenius image gives $b^{q^2} = b$, and hence $b \in F$. Consequently

$$5b = 1, \quad 5b^{-1} = 0.$$

The first equation is impossible in characteristic five; the second is impossible otherwise.

Case $T(u) = 0$. In (21), multiply the first coordinate by u , put $h = y + ut \in H$, and subtract the new first coordinate from the second. This yields

$$\{(h, h^{q^4} + tn) : h \in H, t \in F\}, \quad n = -(u + u^{q^4}). \quad (25)$$

The scaling and subtraction form an ambient $\mathrm{GL}_2(K)$ map, while $(y, t) \mapsto (h, t)$ is an F -linear reparametrization. Here $T(n) = 0$. Also $n \neq 0$: the equality $u^{q^4} = -u$ would imply $u^{q^2} = u$ and hence $u \in F$. In odd characteristic, this forces $u = 0$; in characteristic two it contradicts $T(u) = u \neq 0$. Theorem 4.5 excludes (25). $\square$

5.2. The (C4) family

Proposition 5.2. *No member of the family (C4) is maximum scattered.*

Proof. Let $P(X) = 7X^2 - 5X + 1$ and write $\tau(x) = x^q$. The Ore factorization [16]

$$F_k = (\tau + kI)(I + \tau^2)\tau \quad (26)$$

is verified in Appendix B and will be used in both characteristics. If $k = 1$, then $F_1 = T - I$, so $F_1 = -I$ on the four-dimensional space $H = \ker T$; this is not scattered. Assume $k \neq 1$ and set

$$a = k - 1, \quad s = a^{-1}, \quad N(s + 1) = N(s). \quad (27)$$

Trace adjunction preserves scatteredness by Lemma 2.2(ii).

Odd characteristic. Let $G = \tau + \tau^3$. It is invertible, with

$$G^{-1}(y) = \frac{T(y)}{2} - y - y^q. \quad (28)$$

A direct operator check is given in Appendix B. Parametrizing by $y = G(x)$ and replacing (X, Y) with $(X + Y, Y)$ gives

$$Z = (k - 1)y + \frac{T(y)}{2}, \quad Y = y^q + ky. \quad (29)$$

Here $x \mapsto y$ changes the parameter and the coordinate replacement is an ambient shear. Set

$$D = 2 + T(s), \quad n = s^q + s + 1. \quad (30)$$

Case $D \neq 0$. With $w = sZ$, inversion of the first map in (29) gives

$$T(w) = \frac{D}{2}T(y), \quad y = w - \frac{s}{D}T(w).$$

The shear $Y \mapsto Y - (k/a)Z$ then gives

$$f_c(w) = w^q - cT(w), \quad c = \frac{s^q + s + 1}{2 + T(s)}. \quad (31)$$

If the numerator vanished, then $s^{q^2} = s$, so $s \in F$, $s = -1/2$, and $k = -1$, contrary to $N(k) = 1$.

If F_k were scattered, Theorem 4.1 would give $c \in F$ and $P(c) = 0$. Equation (31) gives $s^{q^2} = s$, hence $s, k \in F$ and $k^5 = 1$. Characteristic five forces $k = 1$. Otherwise

$$c = \frac{k + 1}{2k + 3}, \quad (2k + 3)^2 P\left(\frac{k + 1}{2k + 3}\right) = k^2 + k + 1. \quad (32)$$

The scalar expansion appears in Appendix B. Thus $k^3 = k^5 = 1$, and again $k = 1$, a contradiction.

Case $D = 0$. Put $t = T(y)$ and write $y = h - ts/2$, with $h \in H$. Scaling the first coordinate by s and applying $Y \mapsto Y - k(sZ)$ turns (29) into

$$\{(h, h^q - tn/2) : h \in H, t \in F\}, \quad T(n) = 1. \quad (33)$$

The second projection $u = h^q - tn/2$ has trace $-t/2$ and is bijective. After swapping coordinates, the subspace is the graph of

$$g(u) = u^{q^4} - n^{q^4}T(u).$$

Its trace adjoint is

$$g^*(v) = v^q - T(n^{q^4}v).$$

The matrix $\text{diag}(n^{q^4}, n)$ sends the graph of g^* to the graph of f_n : its first coordinate is $w = n^{q^4}v$, and its second is $w^q - nT(w)$. The adjoint computation is recorded in Appendix B. Thus scatteredness reduces to that of f_n . Theorem 4.1 gives $n \in F$ and $P(n) = 0$. Since $T(n) = 1$, characteristic five is impossible and $n = 1/5$. Since $n = s^q + s + 1$ and $n^q = n$, subtraction gives $s^{q^2} = s$. Hence $s \in \mathbb{F}_{q^2} \cap K = F$, and

$$s = -2/5, \quad k = -3/2.$$

The equation $k^5 = 1$ gives $275 = 0$. Characteristic five has already been excluded, so only characteristic eleven remains; in that characteristic $P(1/5) = 7/25 \neq 0$.

Characteristic two. For $G = \tau + \tau^3$ and $H = \ker T$ one has

$$\ker G = F, \quad \text{im } G = H, \quad G(y + y^q) = y \quad (y \in H).$$

The graph of F_k is therefore parametrized by

$$(x, F_k(x)) = (y + y^q + t, y^q + ky), \quad y \in H, \quad t \in F.$$

This parametrization is bijective: applying G to the first coordinate recovers y , after which t is determined. After adding the two coordinates,

$$Z = (k + 1)y + t, \quad Y = y^q + ky. \tag{34}$$

Here $a = k + 1$ and $s = a^{-1}$. Put

$$D = T(s), \quad n = s^q + s + 1.$$

Case $D \neq 0$. Set $w = sZ = y + st$. Then $t = T(w)/D$, and the shear $Y \mapsto Y + (k/a)Z$ gives

$$f_c(w) = w^q - cT(w), \quad c = \frac{s^q + s + 1}{T(s)}. \tag{35}$$

The numerator is nonzero. If it vanished, then $s^{q^2} = s$, so $s \in F$; substitution in the same equation would give $1 = 0$. If scattered, Theorem 4.1 gives $c \in F$ and $P(c) = c^2 + c + 1 = 0$. Hence $s \in F$, so

$$c = s^{-1} = k + 1, \quad k^5 = 1.$$

Since $P(k + 1) = k^2 + k + 1$, the relations $k^3 = k^5 = 1$ give $k = 1$, contrary to the standing assumption.

Case $D = 0$. Put $h = y + ts \in H$. Scaling the first coordinate by s and applying $Y \mapsto Y + kh$ turns (34) into

$$\{(h, h^q + tn) : h \in H, t \in F\}, \quad T(n) = 1. \quad (36)$$

The trace of the second projection $u = h^q + tn$ is t , so it is bijective. After swapping coordinates we obtain the graph of

$$g(u) = u^{q^4} + n^{q^4}T(u).$$

The trace adjoint is $v \mapsto v^q + T(n^{q^4}v)$. Applying $\text{diag}(n^{q^4}, n)$ to its graph gives the graph of $w \mapsto w^q + nT(w) = f_n(w)$. Hence scatteredness reduces to that of f_n ; the characteristic-two specialization is included in Appendix B. Theorem 4.1 would give $n \in F$ and $P(n) = 0$. On the other hand, $T(n) = 1$ gives $n = 1$, while $P(1) = 1$ in characteristic two. $\square$

Proof of Theorem 1.1. The family (C3) is excluded by Proposition 5.1, and (C4) by Proposition 5.2. The classification then follows from the four-family reduction in [21, Corollary 7.2]. $\square$

Appendix A. Pfaffian and tangent computations

This appendix records the model-specific calculations used in Section 4. Indices are taken modulo five. The Pfaffians follow from (13) and (19) after splitting $K \otimes_F \overline{F}$ into its five embeddings. If deleting row and column i leaves $j < k < \ell < m$, our convention is

$$\text{Pf}_i(\mathcal{A}) = a_{jk}a_{\ell m} - a_{j\ell}a_{km} + a_{jm}a_{k\ell}.$$

In characteristic two, the signs specialize in the usual way; an alternating matrix continues to mean a matrix with zero diagonal.

Appendix A.1. The trace-perturbation section

Relative to $e_0^*, \dots, e_4^*$, the alternating matrix of $\sum_i t_i \ell_i$ in (13) is

$$\mathcal{A}_c(t) = \begin{pmatrix} 0 & (1-c_0)t_0 + c_1t_1 & -c_0t_0 + c_2t_2 & -c_0t_0 + c_3t_3 & -c_0t_0 + (c_4-1)t_4 \\ (c_0-1)t_0 - c_1t_1 & 0 & (1-c_1)t_1 + c_2t_2 & -c_1t_1 + c_3t_3 & -c_1t_1 + c_4t_4 \\ c_0t_0 - c_2t_2 & (c_1-1)t_1 - c_2t_2 & 0 & (1-c_2)t_2 + c_3t_3 & -c_2t_2 + c_4t_4 \\ c_0t_0 - c_3t_3 & c_1t_1 - c_3t_3 & (c_2-1)t_2 - c_3t_3 & 0 & (1-c_3)t_3 + c_4t_4 \\ c_0t_0 + (1-c_4)t_4 & c_1t_1 - c_4t_4 & c_2t_2 - c_4t_4 & (c_3-1)t_3 - c_4t_4 & 0 \end{pmatrix}. \quad (\text{A.1})$$

For $D_c \subset \mathbf{P}_{t_0, \dots, t_4}^4$, the five principal submaximal Pfaffians, indexed by the omitted row and column, are

$$\begin{aligned} Q_0 &= -c_1t_1t_2 - c_1t_1t_3 + c_2t_2t_3 - c_3t_1t_3 + c_4t_1t_4 + c_4t_2t_4 + t_1t_3, \\ Q_1 &= -c_0t_0t_2 - c_0t_0t_3 + c_2t_2t_3 + c_2t_2t_4 - c_3t_3t_4 + c_4t_2t_4 - t_2t_4, \\ Q_2 &= -c_0t_0t_3 + c_1t_1t_3 + c_1t_1t_4 - c_3t_0t_3 - c_3t_3t_4 + c_4t_0t_4 + t_0t_3, \\ Q_3 &= -c_0t_0t_1 + c_1t_1t_4 - c_2t_0t_2 - c_2t_2t_4 + c_4t_0t_4 + c_4t_1t_4 - t_1t_4, \\ Q_4 &= -c_0t_0t_1 - c_0t_0t_2 + c_1t_1t_2 - c_2t_0t_2 \\ &\quad + c_3t_0t_3 + c_3t_1t_3 + t_0t_2. \end{aligned} \quad (\text{A.2})$$

The coordinate vertices lie on D_c . At e_0 , the Jacobian contains the minor $c_0^2c_4$. On $t_0 = t_1 = 0$, the restrictions

$$Q_2 = -c_3t_3t_4, \quad Q_3 = -c_2t_2t_4, \quad Q_0 = c_2t_2t_3 + c_4t_2t_4$$

have common zero set $\{e_2, e_3, e_4\}$.

When $c_0 + c_3 = 1$, the following tangent direction satisfies $Q_1(v) = -c_0^2/c_2$:

$$v = (0, 1, -c_0/c_2, c_0/c_2, 0).$$

If $c_0 + c_3 \neq 1$, a tangent direction is

$$v = \left(0, \frac{c_4(c_0 + c_2 + c_3 - 1)}{c_0(c_0 + c_3 - 1)}, -\frac{c_4}{c_0 + c_3 - 1}, \frac{c_4}{c_0 + c_3 - 1}, 1 \right). \quad (\text{A.3})$$

If $c_0 + c_2 + c_3 - 1 = 0$, then $Q_3(v) = -c_4$. Otherwise substitution in Q_4 gives (14). After $c_i = c$, the same substitution gives (15).

Appendix A.2. The trace-zero vertical section

Use the ordered basis r_0, r_1, r_2, r_3, s and substitute $r_4 = -r_0 - r_1 - r_2 - r_3$. The matrix of $\sum_i z_i \ell_i$ in (19) is

$$\mathcal{A}_n(z) = \begin{pmatrix} 0 & -z_0 - z_1 & -z_0 & -z_0 - z_4 & n_0 z_0 - n_4 z_4 \\ z_0 + z_1 & 0 & -z_2 & -z_4 & n_1 z_1 - n_4 z_4 \\ z_0 & z_2 & 0 & -z_3 - z_4 & n_2 z_2 - n_4 z_4 \\ z_0 + z_4 & z_4 & z_3 + z_4 & 0 & n_3 z_3 - n_4 z_4 \\ -n_0 z_0 + n_4 z_4 & -n_1 z_1 + n_4 z_4 & -n_2 z_2 + n_4 z_4 & -n_3 z_3 + n_4 z_4 & 0 \end{pmatrix}. \quad (\text{A.4})$$

For $D_n \subset \mathbf{P}_{z_0, \dots, z_4}^4$, the principal submaximal Pfaffians are

$$\begin{aligned} Q_0 &= -n_1 z_1 z_3 - n_1 z_1 z_4 + n_2 z_2 z_4 - n_3 z_2 z_3 + n_4 z_2 z_4 + n_4 z_3 z_4, \\ Q_1 &= -n_0 z_0 z_3 - n_0 z_0 z_4 + n_2 z_0 z_2 + n_2 z_2 z_4 - n_3 z_0 z_3 + n_4 z_3 z_4, \\ Q_2 &= -n_0 z_0 z_4 + n_1 z_0 z_1 + n_1 z_1 z_4 - n_3 z_0 z_3 - n_3 z_1 z_3 + n_4 z_1 z_4, \\ Q_3 &= -n_0 z_0 z_2 + n_1 z_0 z_1 - n_2 z_0 z_2 - n_2 z_1 z_2 + n_4 z_1 z_4 + n_4 z_2 z_4, \\ Q_4 &= z_0 z_2 + z_0 z_3 + z_1 z_3 + z_1 z_4 + z_2 z_4. \end{aligned} \quad (\text{A.5})$$

The coordinate vertices lie on D_n , and the Jacobian contains the minor $n_0^2 n_1$ at e_0 . On $z_0 = z_1 = 0$, the equation $Q_4 = z_2 z_4$ confines the common zero set to the union of the coordinate lines $z_2 = 0$ and $z_4 = 0$; Q_1 and Q_0 then leave only e_2, e_3, e_4 .

With $\delta = n_0 + n_2 + n_3$, a tangent direction at e_0 is

$$v = \left(0, \frac{n_0 + n_2}{n_1}, 1, -1, \frac{\delta}{n_0} \right). \quad (\text{A.6})$$

Substitution in Q_1 gives (20). If $n \in F$ and $\text{char } F = 5$, the direction is proportional to $(0, 2, 1, -1, 3)$ and $Q_4(v) = 7 = 2$.

Appendix B. Operator and scalar identities for (C_4)

The factorization (26) follows by composition:

$$(\tau + kI)(I + \tau^2)\tau = (\tau + kI)(\tau + \tau^3) = \tau^2 + \tau^4 + k\tau + k\tau^3.$$

Since $k = (s+1)/s$, the norm condition $N(k) = 1$ is equivalent to $N(s+1) = N(s)$ in (27).

In odd characteristic, put $G = \tau + \tau^3$. For $R(y) = T(y)/2 - y - y^q$,

$$G(R(y)) = T(y) - (y^q + y^{q^2} + y^{q^3} + y^{q^4}) = y,$$

which verifies (28). If $s, k \in F$ and $k = (s + 1)/s$, then

$$c = \frac{s^q + s + 1}{2 + T(s)} = \frac{k + 1}{2k + 3}.$$

Direct expansion gives the second identity in (32). Thus $P(c) = 0$ implies $k^2 + k + 1 = 0$ whenever $2k + 3 \neq 0$.

If

$$g(u) = u^{q^4} - n^{q^4}T(u),$$

then its adjoint under $T(xy)$ is

$$g^*(v) = v^q - T(n^{q^4}v).$$

Writing $w = n^{q^4}v$ and scaling the output by n gives $w^q - nT(w)$. In characteristic two, the same calculation, with plus signs, gives $w^q + nT(w) = w^q - nT(w)$. These are the adjoint reductions used in (33) and (36).

The two exceptional scalar calculations used above are

$$(-3/2)^5 = 1 \implies 275 = 0, \quad P(1/5) = 7/25.$$

After characteristic five has been excluded, the first equality leaves characteristic eleven, where the second expression is nonzero. In characteristic two, $P(X) = X^2 + X + 1$ and $P(1) = 1$.

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Data and code availability

No datasets were generated or analyzed during this study. Reproducibility code and a static verification report are supplied as supplementary material.

References

- [1] A. Blokhuis, M. Lavrauw, Scattered spaces with respect to a spread in $\text{PG}(n, q)$, *Geometriae Dedicata* 81 (1–3) (2000) 231–243. doi:10.1023/A:1005283806897.
- [2] G. Lunardon, O. Polverino, Blocking sets and derivable partial spreads, *Journal of Algebraic Combinatorics* 14 (1) (2001) 49–56. doi:10.1023/A:1011265919847.
- [3] G. Lunardon, O. Polverino, Translation ovoids of orthogonal polar spaces, *Forum Mathematicum* 16 (5) (2004) 663–669. doi:10.1515/form.2004.029.
- [4] G. Lunardon, G. Marino, O. Polverino, R. Trombetti, Maximum scattered linear sets of pseudoregulus type and the Segre variety $\mathcal{S}_{n,n}$, *Journal of Algebraic Combinatorics* 39 (4) (2014) 807–831. doi:10.1007/s10801-013-0468-3.
- [5] B. Csajbók, C. Zanella, On scattered linear sets of pseudoregulus type in $\text{PG}(1, q^t)$, *Finite Fields and Their Applications* 41 (2016) 34–54. doi:10.1016/j.ffa.2016.04.006.
- [6] B. Csajbók, C. Zanella, On the equivalence of linear sets, *Designs, Codes and Cryptography* 81 (2) (2016) 269–281. doi:10.1007/s10623-015-0141-z.
- [7] B. Csajbók, G. Marino, O. Polverino, Classes and equivalence of linear sets in $\text{PG}(1, q^n)$, *Journal of Combinatorial Theory, Series A* 157 (2018) 402–426. doi:10.1016/j.jcta.2018.03.007.
- [8] C. Zanella, F. Zullo, Vertex properties of maximum scattered linear sets of $\text{PG}(1, q^n)$, *Discrete Mathematics* 343 (5), article 111800, 14 pp. (2020). doi:10.1016/j.disc.2019.111800.

- [9] M. Lavrauw, G. Van de Voorde, On linear sets on a projective line, *Designs, Codes and Cryptography* 56 (2–3) (2010) 89–104. doi:10.1007/s10623-010-9393-9.
- [10] B. Csajbók, C. Zanella, Maximum scattered $\mathbb{F}_q$ -linear sets of $\text{PG}(1, q^4)$, *Discrete Mathematics* 341 (1) (2018) 74–80. doi:10.1016/j.disc.2017.07.001.
- [11] G. Longobardi, Scattered polynomials: an overview on their properties, connections and applications, *The Art of Discrete and Applied Mathematics* 9 (2) (2026) #P2.07, 25 pp., published online 22 May 2026. doi:10.26493/2590-9770.1842.2f0.
- [12] P. Delsarte, Bilinear forms over a finite field, with applications to coding theory, *Journal of Combinatorial Theory, Series A* 25 (3) (1978) 226–241. doi:10.1016/0097-3165(78)90015-8.
- [13] E. M. Gabidulin, Theory of codes with maximum rank distance, *Problems of Information Transmission* 21 (1) (1985) 1–12.
URL <https://www.mathnet.ru/eng/ppi967>
- [14] J. Sheekey, A new family of linear maximum rank distance codes, *Advances in Mathematics of Communications* 10 (3) (2016) 475–488. doi:10.3934/amc.2016019.
- [15] B. Csajbók, G. Marino, O. Polverino, F. Zullo, Maximum scattered linear sets and MRD-codes, *Journal of Algebraic Combinatorics* 46 (3–4) (2017) 517–531. doi:10.1007/s10801-017-0762-6.
- [16] Ø. Ore, On a special class of polynomials, *Transactions of the American Mathematical Society* 35 (3) (1933) 559–584. doi:10.1090/S0002-9947-1933-1501703-0.
- [17] B. Wu, Z. Liu, Linearized polynomials over finite fields revisited, *Finite Fields and Their Applications* 22 (2013) 79–100. doi:10.1016/j.ffa.2013.03.003.
- [18] C. Zanella, A condition for scattered linearized polynomials involving Dickson matrices, *Journal of Geometry* 110 (3), article 50, 9 pp. (2019). doi:10.1007/s00022-019-0505-z.
- [19] G. McGuire, J. Sheekey, A characterization of the number of roots of linearized and projective polynomials in the field of coefficients, *Finite Fields and Their Applications* 57 (2019) 68–91. doi:10.1016/j.ffa.2019.02.003.

- [20] B. Csajbók, Scalar q -subresultants and Dickson matrices, *Journal of Algebra* 547 (2020) 116–128. doi:10.1016/j.jalgebra.2019.10.056.
- [21] S. Lia, G. Longobardi, C. Zanella, Towards the classification of maximum scattered linear sets of $\text{PG}(1, q^5)$, *Algebraic Combinatorics* 9 (2) (2026) 327–355. doi:10.5802/alco.483.
- [22] B. Csajbók, G. Marino, O. Polverino, A Carlitz type result for linearized polynomials, *Ars Mathematica Contemporanea* 16 (2) (2019) 585–608. doi:10.26493/1855-3974.1651.e79.
- [23] M. Montanucci, C. Zanella, A class of linear sets in $\text{PG}(1, q^5)$, *Finite Fields and Their Applications* 78, article 101983, 28 pp. (2022). doi:10.1016/j.ffa.2021.101983.
- [24] M. De Boeck, G. Van de Voorde, The weight distributions of linear sets in $\text{PG}(1, q^5)$, *Finite Fields and Their Applications* 82, article 102034, 52 pp. (2022). doi:10.1016/j.ffa.2022.102034.
- [25] G. G. Grimaldi, S. Gupta, G. Longobardi, R. Trombetti, A geometric characterization of known maximum scattered linear sets of $\text{PG}(1, q^n)$, arXiv:2405.01374v1 (2024). doi:10.48550/arXiv.2405.01374.
- [26] A. Cafure, G. Matera, Improved explicit estimates on the number of solutions of equations over a finite field, *Finite Fields and Their Applications* 12 (2) (2006) 155–185. doi:10.1016/j.ffa.2005.03.003.
- [27] K. Slavov, An application of random plane slicing to counting $\mathbb{F}_q$ -points on hypersurfaces, *Finite Fields and Their Applications* 48 (2017) 60–67. doi:10.1016/j.ffa.2017.07.002.
- [28] D. Bartoli, A. Giannoni, G. Marino, A. Neri, An infinite family of non-extendable MRD codes, arXiv:2603.27748v2 (2026). doi:10.48550/arXiv.2603.27748.
- [29] A. Giannoni, The extremal gap for scattered q -systems, arXiv:2607.23156v1 (2026). doi:10.48550/arXiv.2607.23156.
- [30] E. Shinder, Z. Zhang, L -equivalence for degree five elliptic curves, elliptic fibrations and K3 surfaces, *Bulletin of the London Mathematical Society* 52 (2) (2020) 395–409. doi:10.1112/blms.12339.

- [31] D. A. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *American Journal of Mathematics* 99 (3) (1977) 447–485. doi:10.2307/2373926.
- [32] T. Fisher, Pfaffian presentations of elliptic normal curves, *Transactions of the American Mathematical Society* 362 (5) (2010) 2525–2540. doi:10.1090/S0002-9947-09-04876-4.
- [33] K. Hulek, *Projective Geometry of Elliptic Curves*, no. 137 in *Astérisque*, Société mathématique de France, Paris, 1986. doi:10.24033/ast.931.
- [34] A. Weil, *Sur les courbes algébriques et les variétés qui s’en déduisent*, Vol. 1041 of *Actualités Scientifiques et Industrielles*, Hermann, Paris, 1948.
- [35] H. Stichtenoth, *Algebraic Function Fields and Codes*, 2nd Edition, Vol. 254 of *Graduate Texts in Mathematics*, Springer, 2009. doi:10.1007/978-3-540-76878-4.
- [36] B. Csajbók, A. Siciliano, Puncturing maximum rank distance codes, *Journal of Algebraic Combinatorics* 49 (4) (2019) 507–534. doi:10.1007/s10801-018-0833-3.
- [37] A. Neri, P. Santonastaso, F. Zullo, Extending two families of maximum rank distance codes, *Finite Fields and Their Applications* 81, article 102045, 31 pp. (2022). doi:10.1016/j.ffa.2022.102045.